Generalization of the Aoki-Yoshikawa sectoral productivity model based on extreme physical information principle☆

Ilona Bednarek [a], Marcin Makowski [b], Edward W. Piotrowski [b], Jan Sładkowski [a] [1], Jacek Syska [a] [2]

[a] *Institute of Physics, University of Silesia, Uniwersytecka 4, Pl 40-007 Katowice, Poland*

[b] *Institute of Mathematics, University of Bialystok, Bialystok, Poland*

Abstract

This paper presents a continuous variable generalization of the Aoki-Yoshikawa sectoral productivity model. Information theoretical methods from the Frieden-Soffer extreme physical information statistical estimation methodology were used to construct exact solutions. Both approaches coincide in first order approximation. The approach proposed here can be successfully applied in other fields of research.

Keywords: Sectorial productivity, Aoki-Yoshikawa model, Econophysics

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☆Highlights:
Assumptions of original Aoki-Yoshikawa sectoral productivity model (AYM) are given.
Information channel capacity for the AYM in the extremal physical information (EPI) method is constructed.
Analytical observed structural principle and variational one are found.
Generating equation for AYM and probability distribution for AYM are found.
Results of the original AYM and of the EPI method approach to AYM are compared.

Email addresses: ilona.bednarek@us.edu.pl (Ilona Bednarek [a]), makowski.m@gmail.com (Marcin Makowski [b]), qmgames@gmail.com (Edward W. Piotrowski [b]), jan.sladkowski@us.edu.pl (Jan Sładkowski [a]), jacek.syska@us.edu.pl (Jacek Syska [a])

[1]corresponding author

[2]corresponding author

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1. Introduction

Much of economic theory is currently discussed in terms of mathematical economic models. Mathematical economics aims at representation and analysis of problems in economics in order to form meaningful and testable propositions about complex issues often described in a less formal way in everyday life. Econophysics, on the other hand, originates from attempts at solving problems in economics with tools developed by physicists, and is evolving into an interdisciplinary research field. Several applications of the approach to stylized models of economics have been recently put forward [1, 2]. The aim of this paper is to show how the extremal physical information (EPI) method of Frieden and Soffer [3, 4] can be used to develop a generalization of the Aoki-Yoshikawa sectoral productivity model (AYM) [5, 1]. Below, its modified [6, 7] version, which abandons the previous, arbitrary metrical form is presented and the solution to entailed equations of the fully analytical formulation of the information principles problem [6] is given. The approach is based on the maximum-likelihood estimation (MLE) and the Fisher information, both the observed and expected ones, defined as the expectation value of the observed information widely used in information geometry [8] and statistics [4, 7, 9, 6]. A similar approach was previously used to analyse the problem of subjectivity in supply-demand related issues [10, 9]. This paper is organized as follows. In Section 2 the original formulation of the AYM sectoral productivity model is presented. In Section 4 generalization of the method based on the EPI method is introduced. In Section 6 both approaches are compared and, finally, in Section 7 conclusions are drawn.

2. The Aoki-Yoshikawa Sectoral Productivity Model

Sectoral productivity models form key issues in the analysis of productivity growth at an intermediate level of aggregation [5]. Such analyses aim to describe the patterns of productivity growth across and within sectors (e.g. agriculture, manufacturing, and services) and to identify main policy factors driving these patterns. The natural way of presenting these models is in terms of transition probabilities over occupation states. In such an approach, occupation vectors and partition vectors can be given an interpretation in terms of economic variables. In a mathematical approach, irreducible and aperiodic Markov chains that have a unique invariant distribution are used [1, 11]. Such an approach makes it possible to cope with a large number of interacting heterogeneous agents and to some extent ignores the issue of rationality of agents' behaviour as it is impossible to

follow the "motion" of an individual agent in a system composed of about 10^6 individuals. Therefore, the assumption that precise behaviour of each agent is irrelevant, is the crucial point. This enables one to adopt some techniques used in statistical physics and it follows that some models of macroeconomics can be built on analogous premises. In their book [5], Masanao Aoki and Hiroshi Yoshikawa presented, among others, an interesting model for the economy of a country with g economic sectors. The ith sector is characterized by the amount of production factor n_i, that is, the number of workers in the sector i, and by its level of productivity (effectiveness) a_i.

Aoki and Yoshikawa were interested in finding the probability distribution of the productivity among sectors. In the statistical physics language this means that the probability distribution of the occupation vector

$$\vec{n} = (n_1, n_2, ..., n_g) \tag{1}$$

of the system is searched. This coincides with the standard statistical physics problem of finding allocations of n particles to g energy levels. According to Boltzmann, the probability distribution of the occupation vector is equal to [5]

$$\pi(\vec{n}) = \frac{n!}{\prod_{i=1}^{g} n_i!} \prod_{i=1}^{g} p^{n_i} = \frac{n!}{\prod_{i=1}^{g} n_i!} \; p^n \; , \tag{2}$$

where p is the probability of the occupation of the particular sth sector ($s = 1, 2, ..., g$) by the ith worker taken to be the same for all particular configurations of these occupations.

The total production factor is exogenously given and fixed so that

$$\sum_{i=1}^{g} n_i = n \; , \quad (n \; \texttt{fixed}). \tag{3}$$

The output of the i-th sector is given by

$$z_i = a_i \, n_i . \tag{4}$$

The Gross Domestic Product (GPD), that is, the total output of the country Z is equal to

$$Z = \sum_{i=1}^{g} z_i = \sum_{i=1}^{g} a_i \, n_i \; . \tag{5}$$

In the model it is equal to exogenous aggregated demand D, i.e.:

$$Z = D \; , \quad (D \; \texttt{fixed}). \tag{6}$$

Note. It is possible to consider versions of AYM where the demand bound (6) or the condition of constancy of number of workers (3) in the system are relaxed [11].

The standard Lagrange multipliers method can be used to find the occupation vector $\vec{n}$ which maximizes the probability $\pi(\vec{n})$ with conserved both the total production factor n and the total GPD equal to D. With the help of the Stirling formula $\ln(\prod_{i=1}^{g} n_i!) = \sum_{i=1}^{g} n_i(\ln n_i - 1) \;\; (n >> 1 \;\; \texttt{and} \;\; n_i >> 1)$ the problem is reduced to finding the solution of the system of g equations:

$$\frac{\partial}{\partial n_i}\left[\ln \pi(\vec{n}) + \nu \left(\sum_{i=1}^{g} n_i - n\right) - \beta \left(\sum_{i=1}^{g} a_i n_i - D\right)\right] = 0 \; . \tag{7}$$

The solution has the following form:

$$n_i = n_i^* = e^{\nu} \, e^{-\beta a_i} \; , \quad i = 1, 2, ..., g \; . \tag{8}$$

The constants ν and β are determined by inserting (8) in (3) and (5)-(6). This is the Boltzmann distribution for the system which is in the state of the statistical equilibrium. Scalas and Garibaldi showed [11] that there is a more general solution of the form

$$n_i = n_i^{**} = \frac{1}{e^{-\nu} \, e^{\beta a_i} - c} \; , \quad i = 1, 2, ..., g; \; c \in \mathbb{R} \, , \tag{9}$$

where c is a parameter. Eq.(9) arises when the appropriate Markovian dynamics is taken into account, with the transition probabilities which are tuning, via their dependance on the parameter c, the choice of a new productivity sector for workers leaving their sector [11]. Only in case of $c = 0$ the Aoki and Yoshikawa solution (8) is recovered and the interpretation of the cases when $c \neq 0$ can be found in [11].

With the further assumption [1]:

$$a_i = i \; a_0 \qquad i = 1, 2, ..., g \; , \tag{10}$$

where a_0 denotes the minimal productivity, one gets the most probable vector

$$n_i^* = \frac{n}{r-1}\left(\frac{r-1}{r}\right)^i \; , \tag{11}$$

where $r = \frac{D/n}{a_0}$ is the aggregated demand D per agent divided by the minimal productivity. In the limit $r \gg 1$ one gets

$$n_i^* = \frac{n}{r-1} \left(\frac{r-1}{r} \right)^i \approx (\frac{1}{r} + \frac{1}{r^2}) \, e^{-\frac{i}{r}} \; , \quad i = 1, 2, ... \; ; \quad r \gg 1 \; . \tag{12}$$

The assumption $a_i = i \, a_0$ also allows for a simplified worker dynamics via creation and annihilation of components of the occupation vector but due care must be taken to make sure that n is conserved [11]. Alternatively, in the EPI method approach to the AYM model, the probability distribution $p\,(a)$ of the level a of productivity will be found (see Eq.(74)). If the probability distribution $p\,(t, a)$, which is normalized over space and time $\int_0^T \int_{\mathcal{Y}_a} dt \, da \, p\,(t, a) = 1$, then $p\,(t) = \int_{\mathcal{Y}_a} da \, p\,(t, a)$ represents the probability that the worker is found at the time $(t, t + dt)$ somewhere within measurement productivity space $\mathcal{Y}_a$, which is a set of possible values of productivity. For example, a high $p(t) \, dt$ means that there is a high chance that a worker is found anywhere in the space of productivities at this time. In particle physics this property could be called probabilistic creation [4].

3. Basic information on the information channel capacity

Suppose that the original random variable Y takes vector values $\mathbf{y} \in \mathcal{Y}$ and let the k - dimensional vector parameter θ of the distribution $p(\mathbf{y})$ be the *expected parameter*, i.e., the expectation value of Y:

$$\theta \equiv E(Y) = \int_{\mathcal{Y}} d\mathbf{y} \, p(\mathbf{y}) \, \mathbf{y} \; . \tag{13}$$

Let us now consider the N-dimensional sample $\widetilde{Y} = (Y_1, Y_2, ..., Y_N) \equiv (Y_n)_{n=1}^N$, where every Y_n is the variable Y in the nth population, $n = 1, 2, ..., N$, which is characterized by the value of the vector parameter θ_n. The specific realization of $\widetilde{Y}$ takes the form $y = (\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_N) \equiv (\mathbf{y}_n)_{n=1}^N$, where every datum $\mathbf{y}_n$ is generated from the distribution $p_n(\mathbf{y}_n|\Theta)$ of the random variable Y_n, where the $d = k \times N$ - dimensional vector parameter Θ [12] is given by:

$$\Theta = (\theta_1, \theta_2, ..., \theta_N)^T \equiv (\theta_n)_{n=1}^N \; . \tag{14}$$

The set of all possible realizations y of the sample $\widetilde{Y}$ forms the sample space $\mathcal{B}$ of the system. When the variables Y_n of the sample $\widetilde{Y}$ are independent, then the

expected parameter $\theta_{n'} = \int_{\mathcal{B}} dy\, P(y|\Theta)\, \mathbf{y}_{n'}$ does not influence the *point probability* distribution $p_n(\mathbf{y}_n|\theta_n)$ for the sample index $n' \neq n$. The data are generated in agreement with the point probability distributions, which fulfill the condition:

$$p_n(\mathbf{y}_n|\Theta) = p_n(\mathbf{y}_n|\theta_n) \ , \quad \text{where} \quad n = 1, ..., N \ , \tag{15}$$

and *the likelihood function* $P(y\,|\Theta)$ of the sample $y = (\mathbf{y}_n)_{n=1}^{N}$ is the product:

$$P(\Theta) \equiv P\,(y|\Theta) = \prod_{n=1}^{N} p_n\,(\mathbf{y}_n|\theta_n) \ . \tag{16}$$

The Fisher information matrix: Assume that on $\mathcal{B}$ the d - dimensional statistical model:

$$\mathcal{S} = \{P_\Theta \equiv P(y|\Theta), \Theta \equiv (\theta_i)_{i=1}^{d} \in V_\Theta \subset \Re^d\} \ , \tag{17}$$

is given, i.e. the family of the probability distributions parameterized by d non-random variables $(\theta_i)_{i=1}^{d}$ which are real-valued and belong to the parametric space V_Θ of the parameter Θ, i.e. $\Theta \in V_\Theta \subset \Re^d$. Thus, the logarithm of the likelihood function $\ln P : V_\Theta \rightarrow \Re$ is defined on the space V_Θ.

Let $\tilde{\Theta} \equiv (\tilde{\theta}_i)_{i=1}^{d} \in V_\Theta$ be another value of the parameter or a value of the estimator $\hat{\Theta}$ of the parameter $\Theta = (\theta_i)_{i=1}^{d}$. At every point, P_Θ, the $d \times d$ - dimensional observed Fisher information (FI) matrix can be defined [13, 6]:

$$\texttt{iF}(\Theta) \equiv -\,\partial^{i'} \partial^{i} \ln P(\Theta) = \left(-\,\tilde{\partial}^{i'} \tilde{\partial}^{i} \ln P(\tilde{\Theta})\right)_{|_{\tilde{\Theta}=\Theta}} \tag{18}$$

and $\partial^i \equiv \partial/\partial\theta_i$, $\tilde{\partial}^i \equiv \partial/\partial\tilde{\theta}_i$, $i, i' = 1, 2, .., d$. It characterizes the local properties of $P(y|\Theta)$. It is symmetric and in field theory and statistical physics models with continuous, regular and normalized distributions, it is positively definite [13]. We restrict the considerations to this case only. The expected $d \times d$ - dimensional FI matrix on $\mathcal{S}$ at point P_Θ is defined as follows [8]:

$$I_F\,(\Theta) \equiv E_\Theta\,(\texttt{iF}(\Theta)) = \int_{\mathcal{B}} dy P(y|\Theta)\, \texttt{iF}(\Theta) \ , \tag{19}$$

where the differential element $dy \equiv d^N\mathbf{y} = d\mathbf{y}_1 d\mathbf{y}_2 ... d\mathbf{y}_N$. The subscript Θ in the expected value signifies the true value of the parameter under which the data y are generated. The FI matrix defines on $\mathcal{S}$ the Riemannian Rao-Fisher metric [8, 12]. Sometimes, due to the probability distribution normalization and the regularity

condition, the $d \times d$ - dimensional observed Fisher information (FI) matrix can be recorded in the "quadratic" form [8]:

$$\mathtt{iF} = \left(\partial^{i'} \ln P(\Theta)\, \partial^{i} \ln P(\Theta) \right) \ . \tag{20}$$

The central quantity of EPI analysis is the information channel capacity I which is the trace of the (expected) Fisher information matrix. Since under above conditions, the observed Fisher information matrix is diagonal $\mathtt{iF}(\Theta) = \mathrm{diag}(\mathtt{iF}_{nn}(\Theta))$, hence the information channel capacity $I(\Theta)$ is equal to:

$$I(\Theta) = \sum_{n=1}^{N} \int_{\mathcal{B}} dy\, P(y|\Theta)\, \mathtt{iF}_{nn}(\Theta) = \int_{\mathcal{B}} dy\, \mathbf{i} \ , \tag{21}$$

where $\mathbf{i} := P(\Theta)\ \sum_{n=1}^{N} \mathtt{iF}_{nn}(\Theta)$ is the *information channel density* [12, 14].

4. Generalization of the Aoki-Yoshikawa Model

The generalization of the AYM presented in this paper consists in considering productivity as a continuous random variable A. The transition from the discrete variable to the continuous one is performed via

$$A = a_i \to a \quad \text{and} \quad p_i = \frac{n_i}{n} \to p(a) \ . \tag{22}$$

As a consequence, the probability distribution function p has to be normalized:

$$\sum_{i=1}^{g} p_i = 1 \to \int_{\mathcal{Y}_a} da\, p(a) = 1 \ , \tag{23}$$

where $\mathcal{Y}_a$ denotes the set of possible values of the productivity. Analogously, the expectation value of the productivity is replaced in the following way

$$\theta_A \equiv \langle A \rangle = \sum_{i=1}^{g} p_i\, a_i \to \int_{\mathcal{Y}_a} da\ p(a)\, a \,, \tag{24}$$

with the constraint

$$\langle A \rangle = D/n \ . \tag{25}$$

In order to find the probability distribution of the level of productivity A the EPI method is used. Below, the forms of the Fisher information will be adopted to estimate the scalar parameter θ_A, (24), [14]:

$$\theta_A \equiv \langle A \rangle = \int_{\mathcal{Y}_a} da \; p(a) \, a \; , \qquad (26)$$

which in this paper is the expectation value of the random variable of the productivity level A. The basic information on the Fisher information is given in Section 3 and the construction of the information channel capacity can be found in [4, 12].

Previous attempt to solve the Aoki-Yoshikawa productivity problem by the EPI method was based on the Frieden-Soffer approach [4]. However, this paper implements only the analytical form of the (modified) observed structural information principle [14].

4.1. The expected Fisher information and the information capacity of the channel (θ_A)

According to the *the main assumption of the EPI method* proposed by Frieden and Soffer, the system itself samples the space of positions (that is, the space of values of productivity A levels). This space is accessible using its Fisherian degrees of freedom [4, 12]. The sample of the EPI method (so-called inner sample [12, 14]) for the AY model is $N = 1$-dimensional[4]. Thus, both the sample space $\mathcal{B}$ and the base space of events are in AY case equivalent to $\mathcal{Y}_a$ [14].

[4]In [3, 4] the condition of the minimal value of the information (kinematical) channel capacity $I \rightarrow min$ is postulated as the one that fixes the value of N in a unique way. However, sometimes the non-minimal values of I are also discussed as they lead to the EPI method's models, which are of a physical significance [3, 4]. Some discussion on this topic can be also found in [14]. For example, from the analogy with the EPI model of momentum distribution presented in [4], for $N > 1$ the nonequilibrium, stationary solutions for the square flow (where the flow is proportional to the root productivity) might be obtained instead. The point is that when choosing the size N of a sample, different classes of the EPI models are obtained. Yet, with the value of N fixed, the particular EPI model is provided by the solution of one or two information principles. In the case of the discussed AY model, the observed structural principle, which is a differential one, and the variational principle are used. The variational information principle is connected with the extremization of the physical information (see Section 4.2). Thus, the condition $I \rightarrow min$ is choosing the type of the model only [4, 7, 15] whereas with the information principles, the particular solution inside this type of the model is found.

As $N = 1$ thus $p(a|\theta_A) \equiv p(a)$ is the *likelihood function for the AY model* and the $d = 1$-dimensional statistical space of for the AY model is as follows:

$$\mathcal{S} = \{p_{\theta_A} \equiv p(a\,|\theta_A),\ \theta_A \in V_{\theta_A} \subset \Re\}\ , \tag{27}$$

where V_{θ_A} is the parameter space of θ_A. For $N = 1$ the information channel capacity I reduces to the Fisher information $I_F(\theta_1) = I_F(\theta_A)$ for $\theta_1 \equiv \theta_A$, the only parameter. The necessary steps leading from the general form of the Fisher information to the one used in this paper are similar as in [14].

The probability amplitude $q_a \equiv q\,(a|\theta_A)$ is defined in the following way [16, 8, 14]:

$$p\,(a\,|\theta_A) = \frac{1}{4}\,q^2\,(a|\theta_A)\ \ . \tag{28}$$

The $N = 1$-dimensionality of the sample means that the rank of the amplitude $q\,(a|\theta_A)$ of the (productivity) field is also equal to 1 [4].

As θ_A is the scalar parameter and the dimension of the sample is equal to $N = 1$ the information channel capacity $I(\theta_A)$ of the *measurement channel* (θ_A) [12] is equal to *the Fisher information* $I_F(\theta_A)$ *of the parameter* θ_A [14]:

$$I(\theta_A) = I_F(\theta_A)\,. \tag{29}$$

$I_F(\theta_A)$ is the information about the unknown parameter θ_A confined in the $N = 1$-dimensional sample for the random variable A.

The *analytical form* of the (expected) Fisher information on parameter θ_A is equal to [14]:

$$\begin{aligned}
I_F(\theta_A) &= \int_{\mathcal{Y}_a} da\ p\,(a\,|\theta_A)\,\mathtt{iF}_a(\theta_A) \\
&= \int_{\mathcal{Y}_a} da\ p\,(a\,|\theta_A)\left(-\,\frac{\partial^2 \ln p\,(a\,|\theta_A)}{\partial\theta_A^2}\right) \\
&= \int_{\mathcal{Y}_a} da\ \left(-\,\frac{\partial^2 p\,(a\,|\theta_A)}{\partial\theta_A^2} + \frac{1}{p\,(a\,|\theta_A)}\left(\frac{\partial p\,(a\,|\theta_A)}{\partial\theta_A}\right)^2\right) \\
&= \int_{\mathcal{Y}_a} da\ \left(-\,\frac{\partial^2 p\,(a\,|\theta_A)}{\partial\theta_A^2} + \left(q_a^{'}\right)^2\right) \\
&= \int_{\mathcal{Y}_a} da\ \left(-\,q_a q_a^{''} + \frac{\partial^2 p\,(a\,|\theta_A)}{\partial\theta_A^2}\right)\ ,
\end{aligned} \tag{30}$$

where Eq.(28) and the denotations $q_a^{'} \equiv \frac{dq_a(\theta_A)}{d\theta_A}$ and $q_a^{''} \equiv \frac{d^2 q_a(\theta_A)}{d\theta_A^2}$ have been used and the index in $\mathtt{iF}_a(\theta_A)$ signifies the dependence of the observed Fisher information on a. In the last equality the relation:

$$\frac{\partial^2 p\,(a\,|\theta_A\,)}{\partial\theta_A^2} = \frac{1}{2}\,(q_a^{'})^2 + \frac{1}{2}\,q_a\,q_a^{''} \tag{31}$$

was also applied.

Due to the normalization, (23):

$$\int_{\mathcal{Y}_a} da\, p(a|\theta_A) = 1 \;, \tag{32}$$

and the regularity condition [13, 14]:

$$\int_{\mathcal{Y}_a} da\; \frac{\partial^2 p\,(a|\theta_A)}{\partial\theta_A^2} = \frac{\partial^2}{\partial\theta_A^2}\int_{\mathcal{Y}_a} da\; p\,(a|\theta_A) = \frac{\partial^2}{\partial\theta_A^2} 1 = 0 \;, \tag{33}$$

the *analytical form* (30) of the Fisher information transforms into the following *metric form* [14]:

$$\begin{aligned} I_F(\theta_A) \;&=\; \int_{\mathcal{Y}_a} da\; p\,(a\,|\theta_A\,)\,\widetilde{\mathtt{iF}}_a(\theta_A) \\ &=\; \int_{\mathcal{Y}_a} da\; \frac{1}{p\,(a\,|\theta_A)}\left(\frac{\partial p\,(a\,|\theta_A\,)}{\partial\theta_A}\right)^2 = \int_{\mathcal{Y}_a} da\; \left(q_a^{'}\right)^2 \;. \end{aligned} \tag{34}$$

Due to Eq.(33) and in accordance with Eq.(29), both the *EPI method form* of the (expected) Fisher information [14] for the AYM and its information channel capacity for the measurement channel (θ_A) are equal to:

$$I(\theta_A) = I_F(\theta_A) = -\int_{\mathcal{Y}_a} da\; q_a(\theta_A) q_a^{''}(\theta_A) \;. \tag{35}$$

The information channel capacity I is the one which enters into the estimation procedure of the EPI *method.* The presented below derivation of the form of amplitude q_a as the self-consistent solution of the information principles consistently uses the analytical form [6, 14] of the structural information principle, and in this respect, it is different from the derivation of the Boltzmann distribution given in [4].

4.2. The information principles and generating equation

In [7] the existence of the (total) *physical information*[5] K

$$K = I + Q \geq 0 \tag{36}$$

was postulated[6]. The choice of the intuitive condition $K \geq 0$ is connected with the *expected structural information principle* of the EPI method:

$$I + \kappa Q = 0 \tag{37}$$

derived for $\kappa = 1$ in [6], where κ is the so-called efficiency coefficient introduced in [4]. The general forms of I and the *structural information* Q are given in [4, 6]. The form of the information principle, which is more fundamental than (37), is the *observed structural information principle* that has the form $\mathrm{qF} + \mathrm{iF} = 0$ [6]. The derivation of the particular form of the observed structural information principle for the AY model is similar to the one for the EPR-Bohm problem [14] so only necessary steps will be presented.

The other information principle of the EPI method is the *variational information principle* [4]:

$$\delta(I + Q) = 0\ . \tag{38}$$

It has to be stressed that it is the (modified) observed structural information principle [3, 4], [14, 15], and not the expected one, which is solved self-consistently together with the variational information principle. Below, both information principles will be constructed and solved in case of the AYM.

Note. To obtain the value of the efficiency coefficient κ, the information principles, i.e., the (modified) observed structural information principle [14, 15] and the variational information principle, together with the physical preconditions which are specific for the model, e.g., some symmetry conditions, have to be solved simultaneously [4]. As a result both, the specific form of Q and the value of κ are obtained [4]. In [4] it is suggested that for the EPI models which have the quantum counterparts, $\kappa = 1$ [14], whereas for the classical models $0 \leq \kappa \leq 1$ [4]. From the below analysis it follows that in the Aoki-Yoshikawa model $\kappa = 1$,

[5] With such general understanding of K, the diversity of the equations of the EPI method is a consequence of diverse preconditions dictated by the investigated phenomenon [3, 4, 17, 6]).

[6] See also [6, 12, 14], where the differences between the Frieden-Soffer original form of the physical information and information principles used in this paper are discussed.

analogously as in the EPI model of the Boltzmann energy distribution [4]. From the analysis presented in [14] it also follows that Frieden's EPI method of "coverage of quantum mechanics" can be constructed by giving the quantum mechanical interpretation to the statistical probability amplitudes. Yet, in the case of the Einstein-Podolsky-Rosen-Bohm problem the quantum character of the amplitudes is provided by the EPI statistical information theory modelling itself [14]. Thus, the amplitudes $q_a(\theta_A)$ of the EPI method for the Aoki-Yoshikawa model could gain the quantum mechanical interpretation if only a reason for the quantization (e.g., as $a_i = i a_0$ in Eq.(10)) of the productivity levels exists, e.g., as in the Einstein model for specific heat [18]. The assumption (10) is used in [1] for the AYM, which is quoted only in order to compare this model with the one obtained via the EPI method in Section 6.

4.2.1. The information principles for the Aoki-Yoshikawa model

The structural information Q [6] in the AYM for the system described by the set of amplitudes q_a [14] is as follows:

$$Q \equiv \frac{1}{4} \int_{\mathcal{Y}_a} da \; q_a^2(\theta_A) \, \mathsf{qF}_a(q_a) \; , \tag{39}$$

where for the simplicity reason the denotation $q_a \equiv q_a(\theta_A) \equiv q(a|\theta_A)$ is used.

Now, the physical information K, (36), in the AYM is as follows [4, 6]:

$$K = I + Q = \int_{\mathcal{Y}_a} da \; k_a(\theta_A) \; , \tag{40}$$

where I is given by Eq.(30). In Eq.(40), $k_a(\theta_A)$ is the *density of the physical information*, which according to Eqs. (30), (39) and (31) is equal to

$$\begin{aligned} k_a(\theta_A) &= -\, q_a q_a^{''} + \frac{\partial^2 p\,(a\,|\theta_A)}{\partial \theta_A^2} + \frac{1}{4}\, q_a^2 \, \mathsf{qF}_a(q_a) \\ &= -\, \frac{1}{2}\, q_a q_a^{''} + \frac{1}{2} (q_a^{'})^2 + \frac{1}{4}\, q_a^2 \, \mathsf{qF}_a(q_a) \\ &= -\frac{1}{2}\, q_a q_a^{''} + \frac{1}{4}\, q_a^2 \, \widetilde{\mathsf{qF}}_a(q_a) \; , \end{aligned} \tag{41}$$

where the *modified observed structural information* $\widetilde{\mathsf{qF}}_a$ used in the EPI method has been introduced [14]:

$$\widetilde{\mathsf{qF}}_a(q_a) := \frac{2}{q_a^2(\theta_A)} (q_a^{'})^2 + \mathsf{qF}_a(q_a) \; . \tag{42}$$

Under the assumption of analyticity of the log-likelihood function $\ln p\,(a\,|\theta_A\,)$, the Taylor expansion $\ln p\left(a\,\middle|\tilde{\theta}_A\right)$ around the true value of θ_A with the use of the denotations $\frac{\partial^2 \ln p(\theta_A)}{\partial \theta_A^2} \equiv \frac{\partial^2 \ln p(\tilde{\theta}_A)}{\partial \tilde{\theta}_A^2}\,|_{\tilde{\theta}_A=\theta_A}$ and $q_a^{'}(\theta_A) \equiv \frac{\partial q_a(\tilde{\theta}_A)}{\partial \tilde{\theta}_A}\,|_{\tilde{\theta}_A=\theta_A}$, leads to the following form of the observed structural information [14]:

$$\mathsf{qF}_a(q_a) = \frac{1}{q_a^2(\theta_A)} 2\, \left(q_a(\theta_A) q_a^{''}(\theta_A) - \, (q_a^{'}(\theta_A))^2 \right) \; . \tag{43}$$

Here the appearance of q_a in the argument of qF_a means that the probability $p(a|\tilde{\theta}_A)$ (and its derivatives) present in qF_a in the derivation [14] of (43), has been replaced by the amplitude q_a (and its derivatives).

In what follows, the forms of the amplitudes q_a that are the solution to the AYM will be searched for among combinations of the exponential functions. Additional assumption that the term with the first derivative $q_a^{'}(\theta_A)$ on the RHS of the above equation cancels with a term in $\mathsf{qF}_a(q_a)$ has been made.

Now, after moving the term $\frac{1}{2}\,(q_a^{'})^2$ in Eq.(43) from the Fisher information part on its RHS to the structural one on its LHS the *modified observed structural information principle* was obtained [14]. (This shift between $\mathsf{qF}_a(q_a)$ and $\widetilde{\mathsf{qF}}_a(q_a)$ is then used in Eq.(41).) Thus, the modified observed structural information principle for the AYM has the following form [14]:

$$- \quad 2\, q_a(\theta_A) q_a^{''}(\theta_A) + q_a^2(\theta_A)\, \widetilde{\mathsf{qF}}_a(q_a) = 0 \; , \tag{44}$$

where

$$\begin{aligned} \widetilde{\mathsf{qF}}_a(q_a) \;&\equiv\; \left(\mathsf{qF}_a(q_a) + \frac{1}{q_a^2(\theta_A)}\, 2\, (q_a^{'}(\theta_A))^2 \right) \\ &=\; \frac{1}{q_a^2(\theta_A)} 2\, q_a(\theta_A) q_a^{''}(\theta_A) \; . \end{aligned} \tag{45}$$

Equation (44) arises purely as a result of analyticity of the log-likelihood function.

The LHS of Eq.(44) is (up to the factor $\frac{1}{4}$) the density of the physical information $k_a(\theta_A)$ given by Eq.(41). This one is the function of the observed structural information $\mathsf{qF}_a(q_a)$ (which at most can be the function of the amplitudes $q_a(\theta_A)$), of the amplitudes themselves $q_a(\theta_A)$ and of their second derivatives.

Comment: In the AYM, the efficiency factor κ is equal to $\kappa = 1$ [4]. This follows from the fact that except for the information principles, no additional differential constraints are put upon the amplitudes q_a. Thus, the presented EPI model is

a pure analytic one [6], similarly in this respect as in the EPR-Bohm problem [14].

Using Eq.(41) the physical information K, (40), takes the following form:

$$\begin{aligned} K &= I + Q \\ &= \int_{\mathcal{Y}_a} da \, \left(-\frac{1}{2}\, q_a q_a^{''} + \frac{1}{4}\, q_a^2(\theta_A)\, \widetilde{\mathtt{qF}}_a(q_a) \right) \; . \end{aligned} \qquad (46)$$

From Eq.(44) the expected structural information principle (see Eq.(37)), for $\kappa = 1$, follows:

$$I + Q = 0 \; , \qquad (47)$$

where $I + Q$ is given by the RHS of Eq.(46).

The differential equation (44) is the first one from the information principles used in the EPI method. The second one presented below is the variational information principle [4, 7, 6, 14].

In order to obtain the variational information principle, we have to transform the physical information K, (46), into the *metric form*, i.e., the one quadratic in $q_a^{'}$. Therefore, after integration by parts, K can be rewritten as follows [14]:

$$K = I + Q = \int_{\mathcal{Y}_a} da \, \left(k_a^{met}(\theta_A) - \frac{\mathrm{C}_a}{2} \right) \; , \qquad (48)$$

where the constant C_a is equal to:

$$\mathrm{C}_a = \Big(q_a\,(\infty)\, q_a^{'}\,(\infty) - q_a\,(a_0)\, q_a^{'}\,(a_0) \Big) \; , \qquad (49)$$

where a_0 is the smallest (absolute) level of the productivity and $k_a^{met}(\theta_A)$ is the *metric form* of density of the physical information:

$$k_a^{met}(\theta_A) = \frac{1}{2}\, q_a^{'2} + \frac{1}{4}\, q_a^2(\theta_A)\, \widetilde{\mathtt{qF}}_a(q_a) \; . \qquad (50)$$

The variational information principle has the form [4, 14]:

$$\begin{aligned} \delta_{(q_a)} K &\equiv \delta_{(q_a)}\,(I + Q) = \\ &= \delta_{(q_a)} \Big(\int_{\mathcal{Y}_a} da \, (\, k_a^{met}(\theta_A) - \frac{\mathrm{C}_a}{2} \,) \Big) = 0 \; . \end{aligned} \qquad (51)$$

The solution of the *variational* problem (51) with respect to q_a is the *Euler-Lagrange equation*:

$$\frac{d}{d\theta_A}\left(\frac{\partial k_a^{met}(\theta_A)}{\partial q_a'(\theta_A)}\right) \;\; = \;\; \frac{\partial k_a^{met}(\theta_A)}{\partial q_a} \; . \tag{52}$$

From this equation and for $k_a^{met}(\theta_A)$ as in Eq.(50), the following differential equation is obtained for every amplitude q_a:

$$q_a^{''} = \frac{1}{2}\frac{d(\frac{1}{2}q_a^2 \widetilde{\mathsf{qF}}_a(q_a))}{dq_a} \; . \tag{53}$$

As $q_a^2(\theta_A)\widetilde{\mathsf{qF}}_a(q_a)$ is explicitly the function of q_a only, the total derivative has replaced the partial derivative over q_a present in Eq.(52). The obtained form of equation (53) differs slightly from the Frieden form [4] and is the same as in [14]. The origin of this difference is the fully analytical form of density of the physical information (41).

The modified observed structural information principle (44) and the variational information principle (51) (from which the Euler-Lagrange equation (53) follows) serve for the derivation of the equation which generates the distribution.

4.2.2. The derivation of the generating equation

Using the relation (53) in Eq.(44), one can obtain:

$$\frac{1}{2}q_a \frac{d(q_a^2 \widetilde{\mathsf{qF}}_a(q_a))}{dq_a} = q_a^2 \widetilde{\mathsf{qF}}_a(q_a) \; . \tag{54}$$

The above equation can be rewritten in a handier form:

$$\frac{2dq_a}{q_a} = \frac{d\left(\frac{1}{2}q_a^2 \widetilde{\mathsf{qF}}_a(q_a)\right)}{\frac{1}{2}q_a^2 \widetilde{\mathsf{qF}}_a(q_a)} \; , \tag{55}$$

from which, after integration on both sides, the following results can be obtained:

$$\begin{aligned} &\frac{1}{2}q_a^2(\theta_A)\widetilde{\mathsf{qF}}_a(q_a) = \alpha^2 \; q_a^2(\theta_A) \\ &\text{hence} \quad \widetilde{\mathsf{qF}}_a(q_a) = 2\,\alpha^2 \; , \end{aligned} \tag{56}$$

where the constant of integration α^2 is a complex number in general. By substituting Eq.(56) into Eq.(53), we obtain the searched for differential *generating equation* for the amplitudes q_a [4]:

$$q_a^{''}(\theta_A) \;\; = \;\; \alpha^2 \; q_a(\theta_A) \quad \text{for} \quad \theta_A \in V_{\theta_A} \; , \tag{57}$$

which is the consequence of both information principles - the structural and variational ones. This result was obtained previously in [4] for the Boltzmann probability distribution but the arrival at the structural information principle is here [14] different and the form of both information principles also differs slightly.

Note. If an explicit dependence of $\widetilde{\mathsf{qF}}_a(q_a)$ on the productivity a is assumed, i.e. $\widetilde{\mathsf{qF}}_a(q_a, a)$, then a wider scope of solutions to the problem (54) is possible, which also includes non-equilibrium solutions [4]. These solutions correspond to the non-equal probability of the occupation of the particular sth sector by the ith worker for all particular configurations of these occupations (contrary to the assumption used in (2)).

5. The probability distribution for the Aoki-Yoshikawa model

5.1. The definition of the variable of the additive fluctuations

The EPI method analysis for the distribution of the level of productivity is in accord with the general approach of Frieden. The displacement X_a, defined as $X_a = A - \langle A \rangle$, is used instead of values of the productivity level A. Thus, the additive partition is performed: $Y_a \equiv A = \langle A \rangle + X_a$ (similarly, as for the Boltzmann distribution in [4]). It can be performed at the level of the information channel capacity, as it was originally proposed in [4] and developed in [12] for the general distribution which is free of necessity to set the requirement for the shift-invariance, or it can be made at the level of the generating equation. The latter possibility has been chosen as in the considered case this is the simple one, i.e.:

$$\mathbf{y}_a \equiv a = \theta_A + \mathbf{x}_a \; , \quad a_0 \leq \mathbf{y}_a \leq \infty \; , \quad \mathbf{x}_a^{min} = a_0 - \theta_A \leq \mathbf{x}_a < \infty \; , \qquad (58)$$

where $X_a = \mathbf{x}_a$ is a particular displacement. The simplifying assumption that the fluctuation of productivity is unbounded from above, is used (compare [4] for the discussion on distribution of the energy fluctuation). Then, the EPI model is built over the space $\mathcal{X}_a$ of the displacements $\mathbf{x}_a$, which in our case is $\Re$ [12].

A simplifying notation now will be introduced:

$$q_{\theta_A}(\mathbf{x}_a) \equiv q(\mathbf{x}_a + \theta_A | \theta_A) = q(a|\theta_A) \; , \qquad (59)$$

which *leaves the whole information on* θ_A *that characterizes* $q(\mathbf{x}_a + \theta_A|\theta_A)$ *in the index of the amplitude* $q_{\theta_A}(\mathbf{x}_a)$ (and similarly for the original distribution $p_{\theta_A}(\mathbf{x}_a) \equiv p(\mathbf{x}_a + \theta_A|\theta_A) = p(a|\theta_A)$).

Now, appealing to the “chain rule” for the derivative:

$$\frac{d}{d\theta_{\mathbf{A}}} = \frac{d(a-\theta_A)}{d\theta_A}\,\frac{d}{d(a-\theta_A)} = -\,\frac{d}{d(a-\theta_A)} = -\,\frac{d}{d\,\mathbf{x}_a} \tag{60}$$

a transfer from the statistical form (57) of the generating equation to its *kinematical form*[7]:

$$\frac{d^2 q_{\theta_A}(\mathbf{x}_a)}{d\,\mathbf{x}_a^2} = \alpha^2\, q_{\theta_A}(\mathbf{x}_a)\ , \tag{63}$$

where $q_{\theta_A}(\mathbf{x}_a)$ is the amplitude of the distribution of the productivity level fluctuation and it was chosen α to be a real constant (see footnote 8).

5.2. The solution of the generating equation

As the amplitude q_{θ_A} is a real one, thus α^2 in Eq.(63) has also to be real. When the value of the fluctuation of the productivity $\mathbf{x}_a$ is not bounded from above, and this condition is realized by $\mathbf{x}_a^{max}$ approaching infinity, then α has to be real. In this case from the normalization of the squared amplitude we get

$$\frac{1}{4}\int_{\mathbf{x}_a^{min}}^{\infty} d\mathbf{x}_a\, q_{\theta_A}^2(\mathbf{x}_a) = \int_{\mathbf{x}_a^{min}}^{\infty} d\mathbf{x}_a\, p_{\theta_A}(\mathbf{x}_a) = 1\ , \tag{64}$$

it follows that the solution of Eq.(63) is purely of an *exponential* character[8] [4]:

$$q_{\theta_A}(\mathbf{x}_a) = B\, \exp\left(-\alpha\, \mathbf{x}_a\right) + C\, \exp\left(\alpha\, \mathbf{x}_a\right)\ , \quad \alpha \in \mathbf{R}_+\ , \tag{65}$$

---

[7] Note: Taking into account that $d\mathbf{x}_a = d\mathbf{y}_a$, which is connected with the fact that parameter θ_A is a constant, we can transfer from the statistical form of the physical information $K = I + Q$ (40)-(41) to its *kinematical form* with the information channel capacity as follows [4, 6, 14]:

$$I = \int_{\mathcal{X}_a} d\mathbf{x}_a\ \left(\frac{dq_{\theta_A}(\mathbf{x}_a)}{d\,\mathbf{x}_a}\right)^2 \tag{61}$$

and the structural information in the form:

$$Q = \int_{\mathcal{X}_a} d\mathbf{x}_a\ \left(\frac{1}{4}\, q_{\theta_A}(\mathbf{x}_a)^2\, \mathsf{qF}_a(q_{\theta_A}(\mathbf{x}_a))\right)\ . \tag{62}$$

[8] The other possibility for α is that it is a purely imaginary number. Then the solution has the trigonometric character [4]. Yet, when $\mathbf{x}_a^{max} \to \infty$ then due to the normalization condition (64), the trigonometric solution is not the allowed one. (The case when the parametric space is finite can lead to the trigonometric solution, as it is in case of the EPR-Bohm problem [4, 14]).

where B and C are real constants. As the normalization condition (64) is defined in the interval $\mathbf{x}_a^{min} \leq \mathbf{x}_a < \infty$ thus, the part of the solution with the positive exponent has to be rejected due to its divergence to infinity. Therefore, the requirement that $\alpha > 0$ leads to $C = 0$.

In summary, the searched form of the amplitude is as follows:

$$q_{\theta_A}(\mathbf{x}_a) = B \exp\left(-\alpha\,\mathbf{x}_a\right) , \quad \alpha \in \mathbf{R}_+ , \quad \mathbf{x}_a^{min} \leq \mathbf{x}_a < \infty . \tag{66}$$

From this and from the normalization condition (64), the constant B is obtained:

$$B = \pm\, 2\sqrt{2\alpha}\, \exp\left(\alpha\,\mathbf{x}_a^{min}\right) . \tag{67}$$

Thus, the final form of the amplitude for $\alpha \in \mathbf{R}_+$ has the form

$$q_{\theta_A}(\mathbf{x}_a) = \pm\, 2\sqrt{2\alpha}\, \exp\left[\alpha\left(\mathbf{x}_a^{min} - \mathbf{x}_a\right)\right] , \quad \alpha \in \mathbf{R}_+ \tag{68}$$

and α is given in units $[1/productivity]$.

The following probability distribution of the fluctuation of productivity level $\mathbf{x}_a$ can be obtained from the amplitude (68):

$$p\left(\mathbf{x}_a\right) = \frac{1}{4}\, q^2\left(\mathbf{x}_a\right) = 2\alpha\, \exp\left[2\alpha\left(\mathbf{x}_a^{min} - \mathbf{x}_a\right)\right] , \quad \alpha \in \mathbf{R}_+ . \tag{69}$$

In accordance with Eq.(58) the following relation holds $a = \theta_A + \mathbf{x}_a$ thus, $da/d\mathbf{x}_a = 1$. Therefore, the distribution of the random variable A has the form:

$$p\left(a\right) = p\left(\mathbf{x}_a\right) \frac{1}{|da/d\mathbf{x}_a|} = 2\alpha\, \exp\left[-2\alpha\left(a - a_0\right)\right] , \quad a_0 \leq a < \infty . \tag{70}$$

Now, as the expectation value of A is equal to[9]:

$$\langle A \rangle \equiv \theta_A = \int_{a_0}^{+\infty} da\, p\left(a\right) a , \tag{71}$$

thus, inserting Eq.(70) into (71), we obtain:

$$2\,\alpha = \left(\langle A \rangle - a_0\right)^{-1} . \tag{72}$$

[9] Let us notice that from Eq.(71) it follows that A is the unbiased estimator of the expectation value $\langle A \rangle$ of the level of productivity, i.e., $\widehat{\langle A \rangle} = A$.

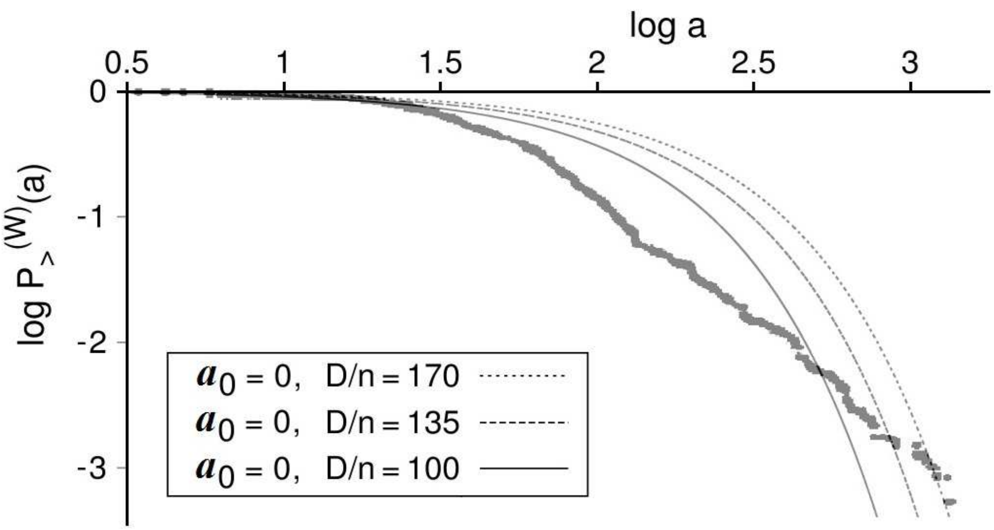

Figure 1: This figure depicts the comparison of the cumulative probability distributions $P_{>}^{(\mathrm{W})}(\mathrm{a}) = \int_{\mathrm{a}}^{\infty} p\,(a)\,da$ [19] calculated for the probability distribution of the level of productivity given by Eq.(74) with the one observed across workers (W) [source [19] for the 2005 year's data] (thick dotted line). The productivity cut "a" is given in the unit of 10^6 yen/person [19] and on both axis the common logarithmic scale is used. The smallest absolute level of the productivity a_0 of the worker is taken to be equal to zero. The ratio D/n (which value is equal to the Boltzmann temperature [19]) has its central value equal to 135 (long dashed line). For comparison, two additional curves, the left one with $D/n = 100$ (solid line) and the right one with $D/n = 170$ (short dashed line) are also plotted [19].

Taking into account the constraint $\langle A \rangle = D/n$, (25), using Eq.(68) finally the amplitude can be obtained:

$$q_{\theta_A}(\mathbf{x}_a) \;=\; \pm\, 2\, \frac{1}{\sqrt{D/n - a_0}}\, \exp\left[-\frac{\mathbf{x}_a + (D/n - a_0)}{2\,(D/n - a_0)}\right] , \qquad \text{where} \quad a_0 - D/n \le \mathbf{x}_a < \infty \tag{73}$$

and the search for probability distribution of the level of productivity:

$$p\,(a) = \begin{cases} \frac{1}{(D/n) - a_0}\ \exp\left(-\frac{a - a_0}{(D/n) - a_0}\right) & \text{for} \quad a \ge a_0 \\ 0 & \text{for} \quad a < a_0 \end{cases} . \tag{74}$$

The distribution (74) is the final result of the EPI method for AY model of productivity.

Finally, Figure 1 shows the comparison of the cumulative probability distributions $P_{>}^{(\mathrm{W})}(\mathrm{a}) = \int_{\mathrm{a}}^{\infty} p\,(a)\, da$ considered in [19] for the probability distribution of the level of productivity given by Eq.(74) with the observed productivity distribution across workers (W) (source [19]). If one considers the whole range of the productivity cut "a" then the exponential law (74) (with the smallest absolute level of the productivity a_0 of the worker equal to zero) fits the data reasonably well.

6. Comparison of two approaches

To compare the approaches, the discrete variable and assumptions have to be reproduced. The assumptions that have been made in the AYM are the following (10):

$$a_i = i\ a_0 \quad \text{where} \quad i = 1, 2, ..., g\ , \tag{75}$$

the number of production sectors is great, $g >> 1$, and

$$r \equiv \frac{D/n}{a_0}\ . \tag{76}$$

This leads to:

$$P(i|\mathbf{n}^*) = \frac{n_i^*}{n} \approx \frac{1}{r-1}\left(\frac{r-1}{r}\right)^i \approx (\frac{1}{r} + \frac{1}{r^2})\ e^{-\frac{i}{r}}\ ,\ i = 1, 2, ...\ ; \ \ r >> 1\ , \tag{77}$$

where n_i^*, $i = 1, 2, ..., g$, are the coordinates of the most probable occupation vector $\vec{n}$ (1). The above equation gives the probability that a randomly selected worker is in the ith sector, provided that the economy is in the state $\vec{n}^* = (n_1^*, n_2^*,$ $..., n_g^*)$.

Therefore, to compare both methods the integration of the obtained probability distribution (74) in the segments (a_i, a_{i+1}) is indispensable. Firstly, let us consider the case for which the "width" of the sectors is equal to $a_0 = a_{min}$, the smallest (absolute) level of the productivity. With this assumption $a_i = i\, a_{min}$, where $a_{min} = a_0 > 0$, (10), the result is obtained:

$$P\,(i) = \int_{ia_0}^{(i+1)a_0} da\ p\,(a) = \left(1 - e^{-1/(r-1)}\right) e^{-(i-1)/(r-1)} \quad \text{for } i = 1, 2, ...\ , \tag{78}$$

which in the limit $r >> 1$ gives:

$$P\left(i\right) \approx (\frac{1}{r} + \frac{1}{2r^2}) \left(e^{-\frac{i}{r}} \; + \; \frac{1}{r}\right) \;\; \text{for} \quad i = 1, 2, ... \;\text{ and }\; a_0 > 0 \;,\; r >> 1 \,. \quad (79)$$

Secondly, let us consider the case of $a_0 = 0$ with the "width" of the sectors equal to δa. This leads to

$$P\left(i\right) = \int_{(i-1)\delta a}^{i\delta a} da \; p\left(a\right) = \left(-1 + e^{1/\tilde{r}}\right) e^{-i/\tilde{r}}, \; i = 1, 2, ... \text{ for } a_0 = 0 \;, \qquad (80)$$

where

$$\tilde{r} \equiv \frac{D/n}{\delta a} \;, \qquad (81)$$

has been introduced instead of r (76). In this case for $\tilde{r} >> 1$ one can finally get:

$$P\left(i\right) \approx \left(\frac{1}{\tilde{r}} + \frac{1}{2\;\tilde{r}^2}\right) e^{-i/\tilde{r}} \;,\; i = 1, 2, ... \;,\; \text{for} \quad a_0 = 0 \;;\; \tilde{r} >> 1 \;. \qquad (82)$$

This means that for large $\tilde{r}$ both methods give the same results in the first order approximation. Note that the solution (80) is exact and in the AYM approach some additional assumptions have to be made to obtain the final formula.

7. Conclusions

The Aoki-Yoshikawa model, although relatively simple, gives interesting results and can be used as a starting point for various analyses. Here we have adopted methods used in theoretical physics to generalize the model and as such these methods refer to the same phenomenon as the original one [5] (see also Figure 1 in Section 5.2 and the discussion in [11]). Our approach allows for exact solutions. The original AY model and the approach presented in this paper agree in first order approximation. Models of phenomena constructed within the proposed approach can be used as tests of the Extreme Physical Information Principle and we envisage successful application of these methods in other fields of research [3, 4, 14].

The original EPI method was invented by Frieden and Soffer [3, 4]. They, together with Plastino and Plastino, put the solution of the (differential) information principles for various EPI models into practice [3]. Nevertheless, the derivation of the generating equation (57) for probability distribution of the level of productivity

(74) (which could have been inferred by the comparison with the Boltzmann distribution [4]) differs from the one used in the original Frieden-Soffer approach [4]. The main difference in the presented derivation is that in this paper the observed physical information used directly in the structural information principle was consistently obtained from the analyticity condition of the log-likelihood function [6, 14] without any jump from its *analytic* to its *metric form*. Only then, the generating equation (57) and (63) was derived. This allows consecutively to obtain the probability distribution (74) of the level of productivity for the statistical informational generalization of the Aoki-Yoshikawa sectoral productivity model.

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